On gradient-like flows on manifolds of dimension four and greater

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A problem

A flow f^t on a smooth closed manifold M^n is called gradient-like if

1. non-wandering set of f^t consists of finite number of hyperbolic equilibria;
2. invariant manifolds of equilibria intersect each other only transversally.

Problems:

- connection of dynamics with topology of ambient manifold;
- topological classification.

Morse inequalities

Smale, 1960-1961: any closed manifold admits a gradient-like flow, and the following theorem is true.

Statement

Let f^t be a gradient-like flow on closed M^n, c_i be a number of equilibria of f^t having unstable manifold of dimension i and $\beta_i = \operatorname{rank} H_i(X)$, $i \in \{0, \dots, n\}$. Then

$$c_0 \geq \beta_0;$$
$$c_1 - c_0 \geq \beta_1 - \beta_i;$$
$$c_2 - c_1 + c_0 \geq \beta_2 - \beta_1 + \beta_0;$$
$$\dots$$
$$\sum_{i=0}^{n} (-1)^i c_i = (-1)^n \chi(M^n).$$

Poincare-Hopf formula (1885, 1926):

$$\sum_p ind_F(p) = \chi(M^n)$$

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in particular, for $M^2 = \underbrace{T^2 \sharp \dots \sharp T^2}_{g}$ and

the flow f^t with k_{f^t} saddles and l_{f^t} nodes,

$$k_{f^t} - l_{f^t} = 2 - 2g$$

Clarification of carrying manifold topology

We will say that a gradient-like flow f^t belongs to the class $G(M^n)$ whenever the following two conditions hold:

(a) Morse index (i.e. dimension of unstable manifold) of a saddle equilibrium state of the flow f^t equals either 1 or $n-1$;

(b) invariant manifolds of distinct saddle equilibria do not intersect each other.

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Theorem (Bonatti, Grines, Medvedev, Pécou, 2002; Grines, G., Pochinka, 2012)

Let $f^t \in G(M^n)$, $n \geq 2$ and $g_{f^t} = (k_{f^t} - l_{f^t} + 2)/2$. Then M^n is homeomorphic either to the sphere $\mathbb{S}^n_0$ if $g_{f^t} = 0$ or to a connected sum $\mathbb{S}^n_{g_{f^t}}$ of $g_{f^t} > 0$ copies of $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

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Theorem (Pilyugin, 1978; Grines, G., Maksimenko, 2021)

Let f^t be gradient-like flow on $\mathbb{S}^n_g$, $g \geq 0$, $n \geq 4$. If invariant manifolds of distinct saddle equilibria of f^t do not intersect each other, then Morse index of each saddle equilibrium equals either 1 or $(n-1)$, that is $f^t \in G(\mathbb{S}^n_g)$.

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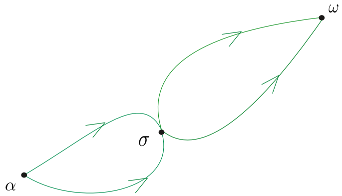

Figure: Closures of invariant manifolds of σ are spheres $S^i = cl\,W^u_\sigma = W^u_\sigma \cup \alpha,\ S^{n-i} = cl\,W^s_\sigma = W^s_\sigma \cup \omega$, homological to zero on $\mathbb{S}^n_g$ for $i \in \{2,\ldots,(n-2)\}$

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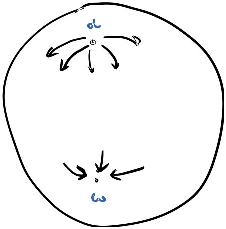

Figure: If non-wandering set of f^t does not contain any saddles then M^n is sphere

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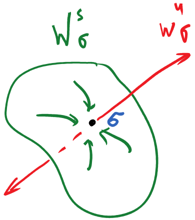

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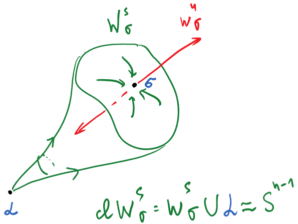

Figure: Suppose non-wandering set of f^t contains a saddle...

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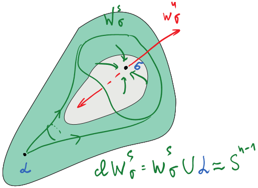

Figure: A trapping nbh $A_\sigma \cong \mathbb{S}^{n-1} \times [-1,1]$ of clW_σ^s

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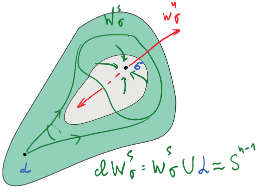

Figure: A trapping nbh $A_\sigma \cong \mathbb{S}^{n-1} \times [-1,1]$ of clW^s_σ

Glue two n-balls to $M^n \setminus int A_\sigma$, and define on obtained manifold M_1 a flow $f_1^t \in G(M_1)$ such that:

- f_1^t coincides with f^t on $M^n \setminus int A_\sigma$;
- f_1^t has a sink on each ball.

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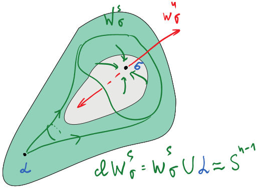

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There are two possibility:

- M_1 is disconnect, then $M_1 = M_{11} \cup M_{12}$ and $M^n = M_{11} \sharp M_{12}$;
- M_1 is connect, then $M = M_1 \sharp \mathbb{S}^{n-1} \times \mathbb{S}^1$ (Medvedev, Umanskii).

Idea of proof

After cutting out all saddles, we get a manifold $M_{k_{f^t}}$ and a flow $f^t_{k_{f^t}}$ with non-wandering set consisting of $(k_{f^t}+l_{f^t})$ sinks and source.
Then $M_{k_{f^t}}$ is disjoint union of $(k_{f^t}+l_{f^t})/2$ spheres, and the number of saddles whose codimension one invariant manifolds cut M^n is $(k_{f^t}+l_{f^t})/2-1$.
So, M^n is homeomorphic to $\underbrace{\mathbb{S}^{n-1}\times\mathbb{S}^1\sharp\dots\sharp\mathbb{S}^{n-1}\times\mathbb{S}^1}_{g}$, where

$$g = k_{f^t} - \big((k_{f^t}+l_{f^t})/2-1\big) = \\ = (k_{f^t}-l_{f^t}+2)/2$$

is the number of saddles whose invariant manifolds do not cut M^n.

History

- $n = 2$: complete classification by Leontovich, Mayer, 1955; Peixoto, 1971; Oshemkov, Sharko, 1998.
- $n = 3$: classification for Morse-Smale flows with finite number of heteroclinic orbits by Y. L. Umanskii, 1990.
- $n \geq 4$: classification in class $G(S^n)$, Pilyugin, 1978.

Peixoto graph (phase diagram)

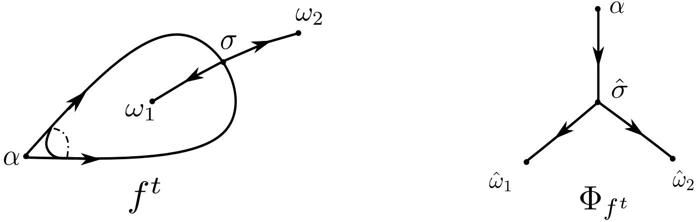
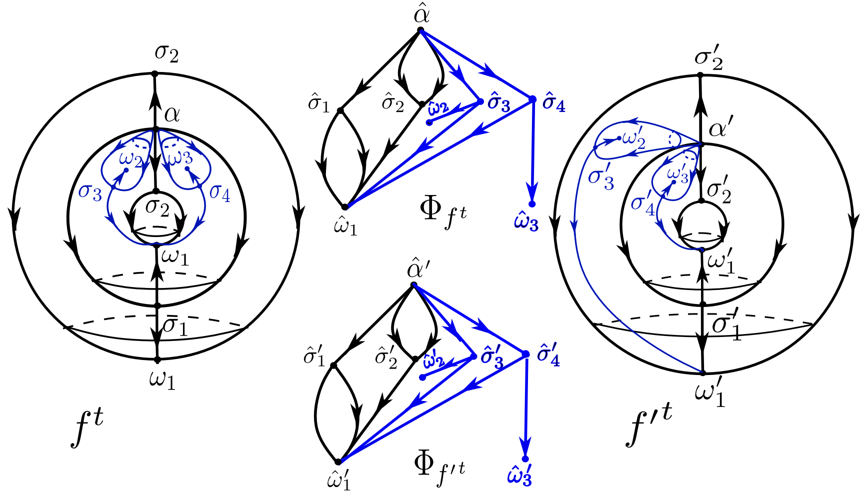


Figure: Phase portrait of a flow $f^t \in G(S^n)$ and its Peixoto graph

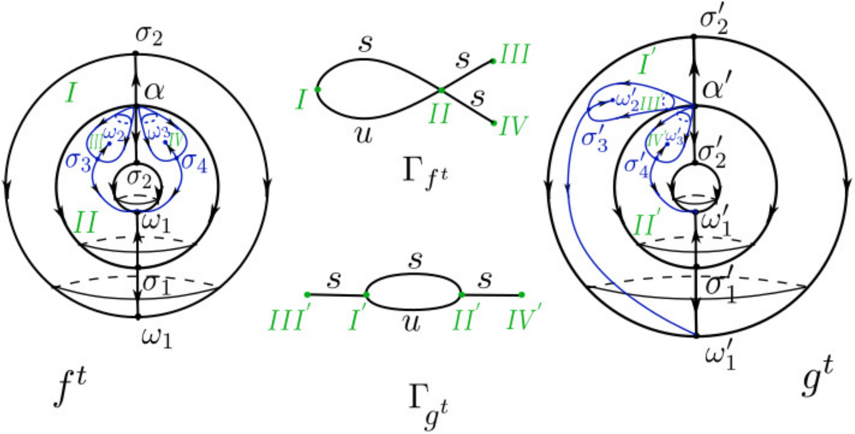
Theorem (Pilyugin, 1978)

Flows $f^t, f'^t \in G(S^n)$ are topologically equivalent iff their Peixoto graphs are isomorphic.

Non-equivalent flows with isomorphic Peixoto graphs on $\mathbb{S}^{n-1} \times \mathbb{S}^1$



Bicolor graph of Oshemkov, Sharko



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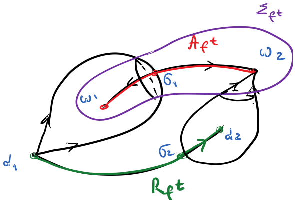

Figure: A section Σ_{f^t}

Theorem (Grines, G.)

Flows $f^t, f'^t \in G(\mathbb{S}^n_g)$ are topologically equivalent iff their bicolor graphs are isomorphic.

Manifolds

Theorem (Grines, Zhuzhoma, Medvedev, 2017-2021)

Let $f^t \in G(M^n)$, $n \geq 3$, l be a numbers of nodes, k — a number saddles of Morse indices $1, n-1$, and m — a number of saddles of Morse indices $1 < j < n-1$. Then $g = (k-l+2)/2$ is non-negative integer and

- *if $g = 0, m = 0$ then $M^n \cong S^n$;*

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- *if $g = 0, m = 0$ then $M^n \cong S^n$;*
- *if $g > 0, m = 0$ then M^n is connected sum $\mathbb{S}^n_g$ of g copies of $S^{n-1} \times S^1$;*

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- *if $g = 0, m > 0$ then there exists a number*

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Impossibility of classification in combinatorial terms

Exception: flows on complex projective plane

Lemma

Let f^t be gradient-like flows without heteroclinic intersection on complex projective plane $\mathbb{C}P^2$. Then f^t has exactly one saddle σ such that $dim\,W^u_\sigma = 2$ and $cl\,W^u_\sigma, cl\,W^s_\sigma$ are locally flat.

Theorem (Grines, G., 2021)

Let f^t, f'^t be gradient-like flows without heteroclinic intersection on complex projective plane $\mathbb{C}P^2$. Then f^t, f'^t are topologically equivalent iff their bicolored graphs $\Gamma_{f^t}, \Gamma_{f'^t}$ are isomorphic.

Idea of proof of Lemm

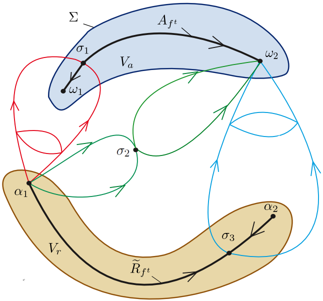

Figure: One-dimensional attractor A_{f^t} and repeller $\widetilde{R}_{f^t}$ of f^t

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