Error of Chernoff approximations based on Chernoff function with a given coefficient at t^2

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This talk is devoted to the error of Chernoff approximations [1,2,3] to strongly continuous one-parameter semigroups [4, 5] in the case when Chernoff function has a coefficient at t^2 which is known.

Let $(X, \|\cdot\|)$ be any Banach space and $\mathscr{L}(X)$ denotes the set of all bounded linear operators on X.

Definition 1 (see, for example, Engel, Nagel [5]). The family $\{G(t)\}_{t\geqslant 0}$ of bounded linear operators on the Banach space X is called the *strongly continuous (one-parameter) semigroup (and also the C_0–semigroup)*, if it is strongly continuous, $G(0) = I$ and for all $t, s \geqslant 0$ the equality $G(t + s) = G(t)G(s)$ is true.

Definition 2 (see, for example: Engel, Nagel [5]). *Generator of a strongly continuous semigroup $\{G(t)\}_{t\geqslant 0}$ on the Banach space* X is the operator $A\colon D(A) \to X$, defined by the equality $Ax = \lim\limits_{t\to+0}(G(t)x - x)/t$ for all x from the domain $D(A)$, where

$$D(A) = \{\ x \in X \mid \lim_{t\to+0}(G(t)x - x)/t \text{ exists }\}.$$

In 1968 Paul Chernoff proved the following theorem:

Theorem 1 (Chernoff [6]). Let X be a Banach space, $F(t)$ be a strongly continuous function from $[0, \infty)$ to a subset of the compressing operators from $\mathscr{L}(X)$, with $F(0) = I$. Suppose that the closure A of the strong derivative $F'(0)$ is the generator of the contracting C_0–semigroup $\{e^{tA}\}_{t\geqslant 0}$. Then $[F(t/n)]^n$ converges to e^{tA} in a strong operator topology.

Let us note that this theorem does not contain an estimate of the rate of convergence, that is, an estimate of the form

$$\|[F(t/n)]^n x - e^{tA}x\| \le C(t, x, n) \to 0 \quad (n \to \infty).$$

In 2022 was published the theorem that provides such estimate under certain conditions:

Theorem 2 (Galkin, Remizov [3]). Suppose that:

1) $T > 0$, $M_1 \ge 1$, $w \ge 0$. $(A, D(A))$ is generator of C_0-semigroup $(e^{tA})_{t\geqslant 0}$ in a Banach space X, such that $\|e^{tA}\| \leqslant M_1 e^{wt}$ for $t \in [0, T]$.

2) There are a mapping $F\colon (0, T] \to \mathscr{L}(X)$ and constant $M_2 \ge 1$ such that we have $\|(F(t))^k\| \leqslant M_2 e^{kwt}$ for all $t \in (0, T]$ and all $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

3) $m \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$, subspace $\mathcal{D} \subset D(A^{m+p})$ is $(e^{tA})_{t\geqslant 0}$-invariant.

4) There exist such functions $K_j\colon (0, T] \to [0, +\infty)$, $j = 0, 1, \ldots, m+p$ that for all $t \in (0, T]$ and all $f \in \mathcal{D}$ we have

$$\left\| F(t)f - \sum_{k=0}^{m} \frac{t^k A^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{m+p} K_j(t)\|A^j f\|.$$

Then: for all $t > 0$, all integer $n \ge t/T$ and all $f \in \mathcal{D}$ we have

$$\|(F(t/n))^n f - e^{tA} f\| \le \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} C_j(t/n)\|A^j f\|,$$

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$C_{m+1}(t) = K_{m+1}(t)e^{-wt} + M_1/(m+1)!$, $C_j(t) = K_j(t)e^{-wt}$ $(j \neq m+1)$.

Let us consider particular example. *Example 1.* Suppose that $\|e^{tA}\| \leq M_1 e^{wt}$, $\|F(t)\| \leq M_2 e^{wt}$, where $w \geq 0$,

$$\|F(t)x - x - tAx\| \leq K_2 t^2 \|A^2 x\|$$

for all $x \in D(A^2)$ and $t \in (0;1]$. Then $m = 1$, $K_0(t) = K_1(t) = 0$ for any $t \in (0;1]$. So theorem 2 states that for any fixed $t > 0$, all $x \in D(A^2)$ and all integer $n \geq t$ the following estimate is true, having the following asymptotic behaviour as $n \to \infty$:

$$\|(F(t/n))^n x - e^{tA}x\| \leq \frac{M_1 M_2 t^2 e^{wt}}{n}\left(K_2 e^{-wt/n} + \frac{M_1}{2}\right)\|A^2 x\| \leq$$
$$\leq M_1 M_2 (K_2 + M_1/2)\frac{t^2 e^{wt}}{n}\|A^2 x\|.$$

So the question arises: what is the lower estimate of the error $\|(F(t/n))^n x - e^{tA}x\|$?

In 2018, Ivan Remizov formulated the following conjecture:

Conjecture 1 (Remizov [7]). Let $(e^{tA})_{t \geq 0}$ be a C_0-semigroup in a Banach space X, and F is a Chernoff function for operator A (recall that this implies $F(0) = I$ and $F'(0) = A$ but says nothing about $F''(0)$) and number $T > 0$ is fixed. Suppose that vector x is from intersection of domains of operators $F'(t)$, $F''(t)$, $F'''(t)$, $F''''(t)$, $F'(t)F''(t)$, $(F'(t))^2 F''(t)$, $(F''(t))^2$ for each $t \in [0, T]$, and suppose that if $Z(t)$ is any of these operators then function $t \to Z(t)x$ is continuous for each $t \in [0, T]$. Then there exists such a number $C_x > 0$, that for each $t \in [0, T)$ and each $n \in \mathbb{N}$ the following inequality holds, where $B = F''(0)$:

$$\|(F(t/n))^n x - e^{tA}x - \frac{t^2}{2n} e^{tA}(B - A^2)x\| \leqslant \frac{C_x}{n^2}.$$

Unfortunately, this hypothesis can only be true if the operators A and B commute. We prove the following theorem:

Theorem 3. Suppose that:

1) C_0-semigroup $(e^{tA})_{t \geqslant 0}$ in a Banach space X has bounded generator $A \in \mathscr{L}(X)$.

2) $T > 0$ and there are a mapping $F\colon [0, T] \to \mathscr{L}(X)$ and constants $M \geq 1$, $w \geq 0$ such that $\|(F(t))^k\| \leqslant M e^{kwt}$ for all $t \in [0, T]$, $k \in \mathbb{N}$.

3) There exist such bounded operator $B \in \mathscr{L}(X)$ and constant $K \geq 0$ that for all $t \in [0, T]$ we have

$$\big\|F(t) - I - tA - \frac{t^2}{2}B\big\| \leqslant K t^3.$$

Then: there exists such a number $C > 0$, that for each $t \in [0, T]$ and each $n \in \mathbb{N}$ the following inequality holds:

$$\big\|(F(t/n))^n - e^{tA} - \frac{t^2}{2n}\int_0^1 e^{tsA}(B - A^2)e^{t(1-s)A}ds\big\| \leq \frac{C}{n^2}.$$

If A and B commute then:

$$\big\|(F(t/n))^n - e^{tA} - \frac{t^2}{2n}e^{tA}(B - A^2)\big\| \leq \frac{C}{n^2}.$$

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